

INJECTIONS INTO FUNCTION SPACES OVER COMPACTA

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ABSTRACT. We study the topology of X given that $C_p(X)$ injects into $C_p(Y)$, where Y is compact. We first show that if C_p over a GO-space injects into C_p over a compactum, then the Dedekind remainder of the GO-space is hereditarily paracompact. Also, for each ordinal τ of uncountable cofinality, we construct a continuous bijection of $C_p(\tau, \{0, 1\})$ onto a subgroup of $C_p(\tau + 1, \{0, 1\})$, which is in addition a group isomorphism.

1. INTRODUCTION

In this paper we continue exploring connections between X and Y given that $C_p(X)$ admits a continuous injection into $C_p(Y)$. We first observe (Theorem 2.4) that if a GO-space X (a subspace of a linearly ordered topological space) admits a continuous injection into $C_p(Y)$, where Y is compact, then its Dedekind remainder is hereditarily paracompact. In other words, the Dedekind remainder of X does not contain a subspace homeomorphic to a stationary subspace of an uncountable regular ordinal. This observation has the same flavor as an earlier result of the author [5, Theorem 2.6], that if τ is an ordinal and X is a subspace of an ordinal such that $C_p(X, \{0, 1\})$ admits a continuous injection into $C_p(\tau, \{0, 1\})$ then $\bar{X} \setminus X$ is hereditarily paracompact. The proof of this earlier statement is rather technical and therefore, it is natural to ask if one can derive the earlier statement from our new one. Clearly, neither is a generalization of the other. To make our new statement usable for derivation of the earlier one we prove that $C_p(\tau, \{0, 1\})$ admits a continuous isomorphism onto a subgroup of $C_p(\tau + 1, \{0, 1\})$ for any ordinal τ of uncountable cofinality (Theorem 2.10). Note that there is no continuous surjection of $C_p(\omega_1, \{0, 1\})$ onto $C_p(\omega_1 + 1, \{0, 1\})$ since the former is Lindelöf and the latter is not. Also, $C_p(\omega_1, \{0, 1\})$ is not homeomorphic to any subspace of $C_p(\omega_1 + 1)$ since the latter has countable tightness and the former does not. Given these observations, the map we construct, even though quite natural, may seem unexpected.

In [4, Theorem 2.9], the author showed that if M is a metric space and $A \subset \omega_1$ then the existence of an injection of $C_p(A, M^\omega)$ into $C_p(\omega_1, M^\omega)$ is equivalent

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to the existence of an embedding of $C_p(A, M^\omega)$ into $C_p(\omega_1, M^\omega)$. The injective map of $C_p(\omega_1, \{0, 1\})$ into $C_p(\omega_1 + 1, \{0, 1\})$ that we construct shows that this statement cannot be extended beyond ω_1 . Namely, it is no longer true even for $\omega_1 + 1$.

In notation and terminology of general topological nature we will follow [6]. For basic facts on $C_p(X)$ we refer to [1]. For basic facts about ordinals, we refer to [7]. Ordinals are endowed with the topology of linear order and their subsets with the subspace topology.

2. STUDY

For convenience we next give a description of the classical construction, Dedekind completion. Even though it can be found in many classical textbooks, we copy it from the author's earlier work [3] since we will use it together with a statement proved in that work.

Dedekind Completion. An ordered pair $\langle A, B \rangle$ of disjoint closed subsets of L is called a *Dedekind section* if $A \cup B = L$, $\max A$ or $\min B$ does not exist, and A is to the left of B . A pair $\langle L, \emptyset \rangle$ ($\langle \emptyset, L \rangle$) is also a Dedekind section if $\max L$ ($\min L$) does not exist. The *Dedekind completion* of L , denoted by cL , is constructed as follows. The set cL is the union of L and the set of all Dedekind sections of L . The order on cL is natural. The order on elements of L is not changed. If $x \in L$ and $y = \langle A, B \rangle \in cL \setminus L$ then x is less (greater) than y if $x \in A$ (B). If $x = \langle A_1, B_1 \rangle$ and $y = \langle A_2, B_2 \rangle$ are elements of $cL \setminus L$, then x is less than y if A_1 is a proper subset of A_2 .

The mentioned statement of our interest follows and will be later used to prove one of our results.

Theorem 2.1. ([3, Corollary 3.5]) *Let L be a GO-space. Then the Dedekind remainder $cL \setminus L$ of L is homeomorphic to a closed subspace of $C_p(L, \{0, 1\})$.*

Before we proceed, let us remind two concepts and a known theorem of C_p -theory. Given a space X , by $e(X)$ we denote the supremum of cardinalities of closed discrete subspaces of X . By $l(X)$ we denote the smallest cardinal number such that every open cover of X contains a subcover of size at most $l(X)$. It is a well-known theorem of Baturov [2] that $e(Z) = l(Z)$ for any subspace Z of $C_p(X)$, where X is a Lindelöf Σ -space (a complete proof can also be found in [1, Theorem III.6.1]). We will use the fact that every Lindelöf locally compact space is a Lindelöf Σ -space (see [9]). The definition of a Lindelöf Σ -space is irrelevant for our discussion, and is therefore, omitted.

Lemma 2.2. *Let S be a stationary subset of a regular uncountable ordinal κ . Suppose that $f : S \rightarrow X$ is a continuous injection. Then there exists a subspace $Z \subset X$ such that $e(Z) \neq l(Z)$.*

Proof. First, observe that the extent of S is less than κ . Indeed, let D be a κ -sized subset of S which is discrete in itself. Denote by D' the derived set of D in κ . That is, $D' = cl_\kappa(D) \setminus D$. The set D' is a closed unbounded subset of κ . Since S is a stationary subset of κ , S meets D' . That is, D is not closed in S . Therefore, any closed discrete subset of S has cardinality less than κ . Since κ is uncountable and regular, we conclude that $e(S) < \kappa$.

Next, observe that $S \setminus \{s\}$ is also stationary, and therefore, has extent strictly less than κ for any $s \in S$. By continuity of f , the inequalities $e(f(S)) < \kappa$ and $e(f(S) \setminus \{x\}) < \kappa$ hold for any $x \in X$. Thus, to prove our lemma it suffices to show that either $f(S)$ or $f(S) \setminus \{x\}$ for some $x \in X$ has Lindelöf number at least κ .

Since S has only one complete accumulation point in βS , we may assume that this point is κ . Let $\tilde{f} : \beta S \rightarrow \beta X$ be the continuous extension to the Čech-Stone compactifications. Since κ is the only complete accumulation point of S and \tilde{f} is continuous, we conclude that $\tilde{f}(S) \setminus U$ is of cardinality strictly less than κ for any neighborhood U of $\tilde{f}(\kappa)$. Since $|f(S)| = \kappa$, we conclude that the pseudocharacter of $\tilde{f}(\kappa)$ in $f(S) \cup \{\tilde{f}(\kappa)\}$ is at least κ . Thus, $Z = f(S) \setminus \{\tilde{f}(\kappa)\}$ has Lindelöf number at least κ and extent strictly less than κ . Hence Z is as desired. \square

Lemma 2.3. *No stationary subset of an uncountable regular ordinal admits a continuous injection into $C_p(X)$, where X is a Lindelöf Σ -space.*

Proof. By Baturov's theorem $l(Z) = e(Z)$ for every $Z \subset C_p(X)$. Now apply Lemma 2.2. \square

Theorem 2.4. *Let L be a GO-space and X a Lindelöf Σ -space. If $C_p(L, \{0, 1\})$ admits a continuous injection into $C_p(X)$ then*

- (1) $cL \setminus L$ does not contain a subspace homeomorphic to a stationary subset of an uncountable regular ordinal; and
- (2) $cL \setminus L$ is hereditarily paracompact.

Proof. Theorem 2.1 and Lemma 2.3 imply (1).

To show (2), we will use a classical theorem of Engelking and Lutzer [8] stating that a GO-space is paracompact iff it contains a closed subspace homeomorphic to a stationary subset of a regular uncountable ordinal. This theorem implies that a GO-space is hereditarily paracompact iff it does not contain a subspace

homeomorphic to a stationary subset of an uncountable regular ordinal. This criterion and (1) imply (2). \square

Observe that if X is a GO-space and has uncountable extent, then $C_p(X)$ does not admit a continuous injection into C_p over a compactum simply because $C_p(X)$ has a subspace homeomorphic to $\{0, 1\}^{\omega_1}$ while C_p over a compactum cannot have such a subspace. This observation and Theorem 2.4 set quite strong requirements on the topology of a GO-space X whose function space admits a continuous injection into the function space over a compactum.

As we mentioned in the introduction, Theorem 2.4 is similar to a particular case of an earlier result of the author that if $C_p(X, \{0, 1\})$ continuously injects into $C_p(\tau, \{0, 1\})$, where X is a subspace of an ordinal and τ is some (other) ordinal, then $\overline{X} \setminus X$ is hereditarily paracompact. Since τ need not be isolated, this statement does not follow from Theorem 2.4. To make the desired reduction for τ of uncountable cofinality we next construct a continuous bijection of $C_p(\tau, \{0, 1\})$ onto a subgroup of $C_p(\tau + 1, \{0, 1\})$, which is, in addition, a group isomorphism. We start with the following definition.

Definition 2.5. *Let τ be an ordinal of uncountable cofinality. For each $f \in C_p(\tau, \{0, 1\})$ we say that $\langle i, \langle i_1, \dots, i_n \rangle \rangle$ is an f -determining sequence if the following conditions are met:*

- (1) $i_1 = 0$,
- (2) $f(0) = i$,
- (3) If $0 < k \leq n$, then $i_k = \min\{\alpha < \tau : i_{k-1} < \tau, f(\alpha) \neq f(i_{k-1})\}$,
- (4) f is constant on $[i_n, \tau)$.

Note that due to continuity of f , the ordinals i_1, \dots, i_n are isolated.

Lemma 2.6. *Let τ be an ordinal of uncountable cofinality and $f \in C_p(\tau, \{0, 1\})$. Then an f -determining sequence exists and is unique.*

Proof. Since τ has uncountable cofinality, there exists the smallest α such that f is constant on $[\alpha, \tau)$. Since $[0, \alpha]$ is a zero-dimensional compactum, there exists a finite partition of $[0, \alpha]$ by convex sets on which f is constant. The left-endpoints of the partition serve as i_1, \dots, i_{n-1} and α as i_n . \square

We next define a correspondence from $C_p(\tau, \{0, 1\})$ to $C_p(\tau + 1, \{0, 1\})$ that will be shown to be a desired map.

Definition 2.7. Let τ be an ordinal of uncountable cofinality, $f \in C_p(\tau, \{0, 1\})$, and $\langle i, \langle i_1, \dots, i_n \rangle \rangle$ the f -determining sequence. Then $\phi(f) : \tau + 1 \rightarrow \{0, 1\}$ is defined as follows:

If $i = 0$, then put:

$$\phi(f)(x) = \begin{cases} 0 & \text{if } x \neq i_2, \dots, i_n \\ 1 & \text{if } x = i_2, \dots, i_n \end{cases}$$

If $i = 1$, then put:

$$\phi(f)(x) = \begin{cases} 0 & \text{if } x \neq i_1, \dots, i_n \\ 1 & \text{if } x = i_1, \dots, i_n \end{cases}$$

Note that $\phi(f)$ is a continuous function from $\tau + 1$ to $\{0, 1\}$ because $\{i_1, \dots, i_n\}$ in the f -determining sequence are isolated ordinals. By Lemma 2.6, $\phi(f)$ is well defined for each f . Therefore, ϕ is a well-defined map from $C_p(\tau, \{0, 1\})$ into $C_p(\tau + 1, \{0, 1\})$.

Lemma 2.8. Let τ be an ordinal of uncountable cofinality. Then $\phi : C_p(\tau, \{0, 1\}) \rightarrow C_p(\tau + 1, \{0, 1\})$ is a continuous injection.

Proof. The conclusion follows from the next two claims.

Claim 1. The map ϕ is one-to-one.

To show that ϕ is one-to-one, fix distinct f and g in $C_p(\tau, \{0, 1\})$. Let $\langle i_f, \langle i_1^f, \dots, i_n^f \rangle \rangle$ and $\langle i_g, \langle i_1^g, \dots, i_m^g \rangle \rangle$ be the f and g -determining sequences. If $\langle i_1^f, \dots, i_n^f \rangle = \langle i_1^g, \dots, i_m^g \rangle$, then f and g are constant on the same clopen intervals. In this case, $f \neq g$ implies that $f(0) \neq g(0)$. That is, $i_f \neq i_g$. We may assume that $i_f = 0$. By the definition of ϕ , we have $\phi(f)(0) = 0$ and $\phi(g)(0) = 1$. Now assume that $\langle i_1^f, \dots, i_n^f \rangle \neq \langle i_1^g, \dots, i_m^g \rangle$. Since the elements of each sequence are in increasing order, there exists an element in one sequence which is not an element of the other. Since $i_0^f = i_0^g = 0$, we may assume that i_3^f is such an element. By the definition of ϕ , we have $\phi(f)(i_3^f) = 1$ and $\phi(g)(i_3^f) = 0$.

Claim 2. ϕ is continuous.

To prove the claim, fix an open U in $C_p(\tau + 1, \{0, 1\})$. We need to show that $\phi^{-1}(U)$ is open. We can assume that U is an element of the standard subbase. That is, there exist $x \in \tau + 1$ and $i_x \in \{0, 1\}$ such that $U = \{g \in C_p(\tau + 1, \{0, 1\}) : g(x) = i_x\}$. We will proceed by exhausting all possibilities on the values of x and i_x .

Case $(x = \tau, i_x = 0)$: In this case U contains all functions that are eventually 0. Since $\phi(f)$ is eventually 0 for every $f \in C_p(\tau, \{0, 1\})$, we conclude that $\phi^{-1}(U) = C_p(\tau, \{0, 1\})$.

- Case $(x = \tau, i_x = 1)$: By the argument of the previous case, $\phi^{-1}(U) = \emptyset$.
- Case $(x = 0, i_x = 0)$: Since $\phi(f)(0) = 0$ if and only if $f(0) = 0$, we conclude that $\phi^{-1}(U) = \{f : f(0) = 0\}$.
- Case $(x = 0, i_x = 1)$: By the argument of the previous case, $\phi^{-1}(U) = \{f : f(0) = 1\}$.
- Case $(x \text{ is limit}, x < \tau, i_x = 0)$: Due to continuity, $f \in C_p(\tau, \{0, 1\})$ cannot change the value at a limit ordinal. Applying the definition of $\phi(f)$, we have $\phi(f)(x) = 0$. Therefore, $\phi^{-1}(U) = C_p(\tau, \{0, 1\})$.
- Case $(x \text{ is limit}, x < \tau, i_x = 1)$: By the argument of the previous case, $\phi^{-1}(U) = \emptyset$.
- Case $(x \text{ is isolated}, 0 < x < \tau, i_x = 0)$: If $f \in \phi^{-1}(U)$, then f does not change its value at x . This means that $f(x) = f(x-1)$. Therefore, $\phi^{-1}(U) = \{f : f(x) = f(x-1)\}$.
- Case $(x \text{ is isolated}, 0 < x < \tau, i_x = 1)$: By the argument of the previous case, $\phi^{-1}(U) = \{f : f(x) \neq f(x-1)\}$.

□

Lemma 2.9. ϕ is an isomorphism of $C_p(\tau, \{0, 1\})$ with its image.

Proof. Fix arbitrary $f, g \in C_p(\tau, \{0, 1\})$. Since ϕ is one-to-one, we only need to show that $\phi(f+g)(x) = (\phi(f) + \phi(g))(x)$ for each $x \in \tau + 1$. This is equivalent to showing the following equality for each $x \in \tau + 1$

$$(*) \quad \phi(f+g)(x) = \phi(f)(x) + \phi(g)(x).$$

We will prove this equality inductively on the value of x . For this let $\langle i_f, \langle i_1^f, \dots, i_n^f \rangle \rangle$ and $\langle i_g, \langle i_1^g, \dots, i_m^g \rangle \rangle$ be the f and g -determining sequences.

Step $x = 0$.

Case $(f(0) \neq g(0))$: Then $\{f(0), g(0)\} = \{0, 1\}$. Hence, $(f+g)(0) = f(0) + g(0) = 1$. By the definition of ϕ , we have $\phi(f+g)(0) = 1$. Thus, the left side of $(*)$ is 1.

Since $\{f(0), g(0)\} = \{0, 1\}$ we obtain that $\{\phi(f)(0), \phi(g)(0)\} = \{0, 1\}$. Therefore, $\phi(f)(0) + \phi(g)(0) = 1$, that is, the right side of $(*)$ is 1 as well.

Case $(f(0) = g(0))$: Then $(f+g)(0) = f(0) + g(0) = 0$. Therefore, the left side of $(*)$ is $\phi(f+g)(0) = 0$.

Since $f(0) = g(0)$, we conclude that $\phi(f)(0) = \phi(g)(0)$. Therefore, the right side of $(*)$ is $\phi(f)(0) + \phi(g)(0) = 0$.

Assumption. Assume that $(*)$ holds for all $x \in [0, \alpha)$, where $\alpha \leq \tau$.

Step $x = \alpha > 0$. Let $k \in \{1, \dots, n\}$ be the largest such that $\alpha \geq i_k^f$. Let $l \in \{1, \dots, m\}$ be the largest such that $\alpha \geq i_l^g$.

Case $(\alpha = i_k^f = i_l^g)$: Then f and g change values at α and hence α is isolated. Therefore, we have $f(\alpha) \neq f(\alpha - 1)$ and $g(\alpha) \neq g(\alpha - 1)$. Then the possibilities for $\langle f(\alpha - 1), g(\alpha - 1), f(\alpha), g(\alpha) \rangle$ are $\langle 1, 1, 0, 0 \rangle$, $\langle 0, 0, 1, 1 \rangle$, $\langle 1, 0, 0, 1 \rangle$, and $\langle 0, 1, 1, 0 \rangle$. In the first two of cases, we have $(f + g)(\alpha - 1) = (f + g)(\alpha) = 0$. In the other two cases, we have $(f + g)(\alpha - 1) = (f + g)(\alpha) = 1$. In all cases we have $(f + g)(\alpha - 1) = (f + g)(\alpha)$. That is, $(f + g)$ does not change its value at α . Therefore, the left side of (*) is $\phi(f + g)(\alpha) = 0$.

To evaluate the right side of (*), observe that the case's condition implies that $\phi(f)(\alpha) = \phi(g)(\alpha) = 1$. Therefore, the right side of (*) is 0 too.

Case $(\alpha = i_k^f \text{ and } i_k^f \neq i_l^g)$: Then f changes its value at α while g does not. Since f changes its value at α , α is isolated. The possibilities for $\langle f(\alpha - 1), f(\alpha) \rangle$ are $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$. The possibilities for $\langle g(\alpha - 1), g(\alpha) \rangle$ are $\langle 0, 0 \rangle$ and $\langle 1, 1 \rangle$. Therefore, the possibilities for $\langle f(\alpha - 1) + g(\alpha - 1), f(\alpha) + g(\alpha) \rangle$ are $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$. That is, $(f + g)$ changes its value at α . Therefore, the left side of (*) is $\phi(f + g)(\alpha) = 1$.

To evaluate the right side of (*), observe that the case's conditions imply that f changes its value at α while g does not. Therefore, $\phi(f)(\alpha) = 1$ and $\phi(g)(\alpha) = 0$. Therefore, the right side of (*) is $\phi(f)(\alpha) + \phi(g)(\alpha) = 1$.

Case $(\alpha = i_l^g \text{ and } i_k^f \neq i_l^g)$: This case is analogous to the previous case.

Case $(\alpha \neq i_k^f \text{ and } \alpha \neq i_l^g)$: In this case, there exists $\beta < \alpha$ such that f and g are constant on $[\beta, \alpha]$. Then $f + g$ is constant on $[\beta, \alpha]$. Therefore, the left side of (*) is $\phi(f + g)(\alpha) = 0$.

Let us evaluate the right side of (*). The case's conditions imply that $\phi(f)(\alpha) = 0$ and $\phi(g)(\alpha) = 0$. Therefore, the right side of (*) is $\phi(f)(\alpha) + \phi(g)(\alpha) = 0$.

□

We can summarize statements 2.6-2.9 as follows:

Theorem 2.10. *Let τ be an ordinal of uncountable cofinality. Then there exists a continuous one-to-one map of $C_p(\tau, \{0, 1\})$ onto a subgroup of $C_p(\tau + 1, \{0, 1\})$, which is, a group isomorphism.*

Note that $C_p(\omega, \{0, 1\})$ does not admit a continuous injection into $C_p(\omega + 1, \{0, 1\})$ since the latter is countable while the former is uncountable. Therefore, the condition on cofinality of τ in our construction of ϕ is important.

Our results can be used to derive some earlier results of the author. Namely, in [5, Theorem 2.6], the author proved that if M is a metric space with at least two elements, τ is an ordinal, and X is a subspace of an ordinal such that $C_p(X, M)$ admits a continuous injection into $C_p(\tau, M)$, then $\overline{X} \setminus X$ is hereditarily paracompact. The results of this paper can be used to derive the mentioned earlier result for the case when $M = \{0, 1\}$. Indeed, Let X be a subspace of an ordinal and let $C_p(X, \{0, 1\})$ admit a continuous injection into $C_p(\tau, \{0, 1\})$ for some ordinal τ . Let κ be the smallest ordinal number such that $X \subset \kappa$. Clearly, $Cl_{\kappa+1}(X)$ is the Dedekind completion of X . If τ is of countable cofinality, then τ is locally compact or compact, that is, a Lindelöf Σ -space. By Theorem 2.4, the Dedekind remainder of X is hereditarily paracompact. If τ has uncountable cofinality, then $C_p(\tau, \{0, 1\})$ admits a continuous injection into $C_p(\tau + 1, \{0, 1\})$, and therefore, $C_p(X, \{0, 1\})$ admits a continuous injection into $C_p(\tau + 1, \{0, 1\})$. Now apply Theorem 2.4 to conclude that $Cl_{\kappa+1}(X) \setminus X$ is hereditarily paracompact. Of course it would be nice if were able to derive the most general version of the earlier result using our new approach. But for this, we need a positive answer to the following question.

Question 2.11. *Let τ be an ordinal of uncountable cofinality and M a metric space containing at least two points. Is it true that $C_p(\tau, M)$ admits a continuous injection into $C_p(\tau + 1, M)$?*

Next are natural questions prompted by the properties of our ϕ defined in Definition 2.7.

Question 2.12. *Let X be a countably compact locally compact space. Is it true that $C_p(X)$ admits a continuous injection into $C_p(Y)$ for some compactum Y .*

Question 2.13. *Let τ be an ordinal of uncountable cofinality and G a topological group. Is it true that $C_p(\tau, G)$ admits a continuous isomorphism onto a subgroup of $C_p(\tau + 1, G)$?*

In [4], the author showed that for a subspace A of ω_1 and a non-trivial metric space M , the existence of an embedding of $C_p(A, M^\omega)$ into $C_p(\omega_1, M^\omega)$ is equivalent to the existence of an injection of $C_p(A, M^\omega)$ into $C_p(\omega_1, M^\omega)$. The results of this paper show that this criterion cannot be extended beyond ω_1 . Indeed, by Lemma 2.8, $C_p(\omega_1, \{0, 1\})$ admits a continuous injection into $C_p(\omega_1 + 1, \{0, 1\})$. Then $C_p(\omega_1, \{0, 1\})^\omega$ admits a continuous injection into $C_p(\omega_1 + 1, \{0, 1\})^\omega$. Since $C_p(X, Y)^\omega$ is homeomorphic to $C_p(X, Y^\omega)$ (see [1, Proposition 0.3.3]), we conclude that $C_p(\omega_1, \{0, 1\}^\omega)$ admits a continuous injection into $C_p(\omega_1 + 1, \{0, 1\}^\omega)$.

However, $C_p(\omega_1, \{0, 1\}^\omega)$ does not embed into $C_p(\omega_1 + 1, \{0, 1\}^\omega)$ since the latter has countable tightness while the former does not. Nonetheless, we believe that the mentioned earlier result may have a chance to be extended to the class of first-countable countably compact subspaces of ordinals.

Question 2.14. *Let X be a countably compact first-countable subspace of an ordinal and Z a subspace of X . Is it true that $C_p(Z)$ continuously injects into $C_p(X)$ iff $C_p(Z)$ embeds into $C_p(X)$? Is it true that $C_p^\omega(Z)$ continuously injects into $C_p(X)^\omega$ iff $C_p(Z)^\omega$ embeds into $C_p(X)$?*

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